Lesson 5. Poisson Arrival Processes – Decomposition and Superposition

0 Warm up

Example 1. A radioactive source emits particles according to a Poisson process with an arrival rate of 2 per minute.

a. What is the probability that the first particle appears some time after 3 minutes but before 5 minutes?
b. What is the probability that exactly one particle is emitted in the interval from 3 to 5 minutes?

1 Overview

- Decomposing a Poisson process into two arrival counting subprocesses
- Superposing (combining) two Poisson processes into one arrival counting process

2 Decomposition of Poisson processes

- Let's think back to Beehunter case
- Accidents (arrivals) occur according to a Poisson process with arrival rate λ accident/week
- Suppose that a fraction $(1 - \gamma)$ of these accidents are major, γ are minor

- We can model each accident type as a **Bernoulli random variable** with success probability γ

$$B = \begin{cases} 0 & \text{with probability } 1 - \gamma \quad \text{(major accident)} \\ 1 & \text{with probability } \gamma \quad \text{(minor accident)} \end{cases}$$

- Let's assume:
  - accident types for all accidents are independent and time stationary
  - accident types and interarrival times are independent
- The **decomposition property**:
  - Type 0 arrivals (e.g. major accidents) form a Poisson process with arrival rate $\lambda_0 = (1 - \gamma)\lambda$
  - Type 1 arrivals (e.g. minor accidents) form a Poisson process with arrival rate $\lambda_1 = \gamma\lambda$
  - These two processes are independent
- This works because the Poisson process is decomposed by a independent Bernoulli variables
- Other methods of decomposition do not necessarily lead to Poisson subprocesses
- Proof on p. 111 of Nelson

Example 2. Suppose that in the Beehunter case, the accident rate is in fact 3/2 per week. In addition, 80% of the accidents are minor, and 20% are major.

a. What is the probability that fewer than 4 minor accidents occur during any 4-week period?
b. What is the expected number of major accidents in any 8-week period?

Example 3. You have been asked to conduct a study of the pedestrian crossing near Chick & Ruth's in Downtown Annapolis. Assume the following behavior. Pedestrians approach the crossing at a rate of 6 pedestrians per minute; one-third of them are on the left side, and two-thirds of them are on the right side. Pedestrians wait until the "WALK" signal, at which time all waiting pedestrians cross instantaneously. Suppose the "WALK" signal occurs every 2 minutes.

a. What is the expected number of pedestrians crossing left to right on a given "WALK" signal?
b. What is the probability that at least one pedestrian crosses right to left on any particular signal?

3 Superposition of Poisson processes

- We can also combine Poisson processes
- Suppose that:
  - major accidents arrive at the intersection according to a Poisson process with arrival rate λ_0
  - minor accidents arrive at the intersection according to a Poisson process with arrival rate λ_1
  - these processes are independent of each other
- The **superposition property**: the arrivals from both processes (e.g. major and minor accidents) together form a Poisson process with arrival rate $\lambda = \lambda_0 + \lambda_1$
- This works because the two Poisson processes are independent
- Proof on pp. 111-112 of Nelson

Example 4. The Bank of Simplexville opens at 8 a.m. Customers arrive at the lobby at a rate of 10 per hour. The bank also has a separate ATM where customers arrive at a rate of 20 per hour. Approximate these arrival processes as Poisson processes. What is the probability that the 100th bank customer (at the lobby or the ATM) arrives before noon, given that 50 customers have arrived by 10 a.m.?

Example 5. The Markov Company has two salespeople, John and Louise. On average, John receives 8 orders per week, while Louise receives 12 per week. Suppose the orders arrive according to a Poisson process.

a. What is the probability that the total sales for two weeks will be more than 30 orders?
b. What is the expected number of orders in one month?

4 Exercises

Problem 1. (Nelson 5.15) This problem concerns capacity planning for a manufacturing company. The company has two salespersons, John and Louise, who each cover one half of the United States. At the end of each week, the salespersons report their sales to the company, which then manufactures the products that have been ordered.

The company has three products, which it calls A, B and C for simplicity. Each salesperson obtains 10 orders per week, on average, of which approximately 20% are for A, 70% are for B, and 10% are for C. In terms of capacity, it takes 25 person-hours to produce one A, 15 person-hours to produce one B, and 40 person-hours to produce one C.

Help the company do its capacity planning by answering the following questions. You may assume that the arrival of orders to each salesperson can be well approximated as a Poisson process.

a. A Poisson process follows the stationary-increments property. What is the physical interpretation of the stationary-increments property in this situation?
b. What is the probability that the total sales for 1 week will be more than 30 products?
c. Capacity can only be changed on a monthly basis. What is the expected number of person-hours the company will need over a 1-month period?
d. What is the probability that Louise will sell more than 5 product Bs on each of 2 consecutive weeks?